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by B. Herzog and E. F. Masur

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SUMMARY

When subjected to sufficiently large radial pressure, a circular plate buckles into a configuration which is governed by a set of nonlinear equations. The present note considers the frequencies and modes of vibration of small amplitude about this configuration. Perturbation techniques are employed near the buckling point; as buckling proceeds solutions are obtained through power series expansions. Although the static buckling configuration is radially symmetric, no such restriction is imposed on the modes of vibration. The analysis is "exact" within the limits of classical plate theory and in the sense of a converging series which has been truncated.

As a by-product of the analysis it is shown, at least within the range of buckling amplitudes considered, that all the frequencies of vibration are real. This suggests that observed secondary buckling phenomena are not traceable to instability of the radially symmetric static configuration.

II. INTRODUCTION, CREDITS, SYMBOLS

Introduction

Modern engineering structures have been experiencing a very rapid decrease in their "thickness" dimension as severe weight (and other) limitations have been imposed during recent years. As a consequence many structures are used in a postbuckled state, in which the loads sustained are greater than those predicted in the usual "Euler column" sense. In addition, structures loaded in this manner are frequently expected to survive an environment of dynamic forces while subjected to these high static loads. The purpose of the present study is to determine the dynamic characteristics of such a structure, that is, the natural frequencies and shapes of the modes of vibration of a circular plate as functions of a load parameter.

If a structure is loaded statically, compressive forces generally tend to decrease the frequency of vibration. The stiffness of the structure is thus reduced, and in fact vanishes when the lowest frequency of vibration reaches zero. In the case of a conservative loading system, this condition is identified with "buckling" in the sense that the equilibrium is no longer stable.

The best known example of such a problem is the lateral vibration of an elastic bar which is axially loaded.[1] The mode shape is sinusoidal for a simply supported bar, and the square of the frequency of vibration is linearly related to the axial force (or an associated loading parameter). Lurie [2] discusses several examples related to vibration and structural stability and cites both theoretical and experimental results. He shows that, in general, within the framework of linear theories, whenever the mode shape of buckling and of vibration in the presence of axial loads is the same, the interaction curve between the square of the frequency and some monotonic increasing load parameter is linear. Massonnet [3] treats the same subject extensively, but selects problems of greater mathematical complexity, which he solves by approximate means, such as the Rayleigh-Ritz method.

The linear equations of the classical theory of plates have been investigated extensively.[4] The buckling of a circular plate was first studied by Bryan.[5] Federhofer [6] and Willers [7] have studied the problem of the vibrating edge plate subjected to edge loads and have presented extensive results of the interaction between compressive (and tensile) forces and the frequency of lateral vibration of the plate.

There exist few exact solutions for the nonlinear equations of buckled plates, which were introduced in 1910 by von Kármán. [8] The problem is particularly difficult for rectangular plates where several approximate methods have been introduced, such as the one by Marguerre. [9] Bisplinghoff and Pian [10] have treated the vibration of a rectangular plate of infinite length and, in some cases, that of plates of finite length. For the circular plate several more solutions are available. Way [11] has solved, by power series methods, the problem of a circular plate subjected to lateral load. Friedrichs and Stoker [12,13] have used perturbation and power series methods to solve the problem of the simply supported circular plate subjected to compressive radial loading in the plane of the plate. The methods of these writers have been applied by Bodner [14] to a clamped edge plate for the same type of loading. Bromberg[15] has used the methods utilized by Friedrichs and Stoker to study the effect of very large lateral loads which give rise to certain instabilities. Keller and Reiss[16] have applied numerical methods to the problem discussed by Friedrichs and Stoker. Similar problems are studied by Alexeev[17] and, as a special case, in a paper by Panov and Feodossiev.[18] Masur [19] has utilized a stress function space to obtain a sequence of solutions with error estimates.

In a recent paper, Massonnet [20] considers the effects of initial curvature on the natural frequencies of vibration of an edge-compressed, clamped edge, circular plate. He solves the static problem by the method of Friedrichs and Stoker and then assumes that the mode shape of vibration is the same as that of the static problem, and utilizing the Rayleigh-Ritz method obtains the approximate frequency of vibration.

The present study is concerned with small vibrations of a circular plate. The plate is radially compressed (as a result of aerodynamic heating or other causes) and buckles into a radially symmetric configuration, such as is treated in (12) and (13). With the edges fixed against additional radial displacements, the buckled plate is now subjected to dynamic lateral loads. If only small additional time-dependent displacements are considered (that is, if the resulting equations are linearized with respect to these additional displacements), such problems can be handled by the usual spectral analysis methods. Information is therefore needed regarding the natural frequencies and modes of vibration. To obtain this information is the object of the present investigation; it may be noted that while the buckled static configuration exhibits radial symmetry, no such restriction is imposed on the dynamic response.

Credits

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Symbols

e_{ij}	components of membrane strains
h	thickness of plate
M_{ij}	components of bending moments, non-dimensional
p	lateral surface load on the plate
t	time
t_{ij}	components of dynamic membrane stress resultants, non-dimensional
u,v	components of membrane displacements
w	lateral displacement of plate
D	flexual rigidity of plate
E	Young's modulus of elasticity of plate material
F	stress function
G	shear modulus of elasticity of plate material
J_n, I_n	Bessel functions
M_{ij}	components of bending moments
N_{ij}	components of membrane stress resultants
T_{ij}	components of static membrane stress resultants, non-dimensional
U	radial component of membrane displacements, non-dimensional
W	static lateral displacement of plate, non-dimensional
γ	parameter influenced by thickness and Poisson's ratio
ϵ	perturbation parameter
ϕ,Φ	stress functions associated with dynamics and statics respectively
ρ	mass density of plate material
λ	membrane loading parameter
μ	vibration frequency parameter
ν	Poisson's ratio of plate material

ω frequency of vibration

∇^2 Laplacian operator in polar coordinates specialized for radial dependence

Δ general two-dimensional Laplacian operator

II. FORMULATION OF THE PROBLEM

In what follows we consider the xy plane to be the middle plane of the plate, and z the direction of the lateral deflection. The plate may be subjected to membrane forces in the plane of the plate and to lateral loads in the z direction. The thickness of the plate is h. In the absence of body forces in the x and y directions the two relevant differential equations (8) of lateral equilibrium and compatibility are

$$D\Delta\Delta\bar{w} - Q(F,\bar{w}) = p \qquad (2.1)$$

$$\Delta\Delta F = -\frac{Eh}{2} Q(\bar{w},\bar{w}) \qquad (2.2)$$

where Δ represents the Laplacian operator, F is the Airy stress function, w is the lateral deflection of the plate and p is the load per unit area applied to the lateral surface of the plate. The flexural rigidity is

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

where E is the Young's Modulus of Elasticity and ν is Poisson's ratio, while the quadratic operator Q is defined by means of

$$Q(f,g) \equiv f_{,xx}g_{,yy} + f_{,yy}g_{,xx} - 2f_{,xy}g_{,xy}$$

with a comma followed by a letter representing appropriate differentiation.

For a moving plate the terms associated with the inertia in the plane of the plate are neglected in comparison to those due to the lateral motion;* the latter are given by

$$p = -\rho h \frac{\partial^2 \bar{w}}{\partial t^2} \qquad (2.3)$$

where ρ is the mass per unit volume.

*For a discussion to this point see references (21) and (22).

Only small amplitude harmonic vibrations with respect to the static configuration of larger amplitude are considered. Consistent with this assumption the following partitioning of the stress function F, the displacements, strains and other quantities is appropriate:

$$
\begin{aligned}
F &= F^S + \epsilon^* F^D e^{i\omega t} \\
\overline{w} &= \overline{w}^S + \epsilon^* \overline{w}^D e^{i\omega t} \\
e_{ij} &= e_{ij}^S + \epsilon^* e_{ij}^D e^{i\omega t} \\
N_{ij} &= N_{ij}^S + \epsilon^* N_{ij}^D e^{i\omega t}
\end{aligned}
\qquad (2.4)
$$

in which e_{ij} and N_{ij} are the cartesian components of the membrane strains and forces, while ω is the frequency of vibration and ϵ^* is **a small parameter.** The membrane forces N_{ij} are related to the stress function F by

$$
N_{ij} = c_{ik}\, c_{jl}\, F_{,kl} \qquad \left(c_{ij} = \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}\right) \qquad (2.5)
$$

and the membrane strains e_{ij} to the forces by

$$
e_{ij} = \frac{1}{Eh}\left[(1+\nu)N_{ij} - \nu N_{kk}\delta_{ij}\right] \qquad \delta_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (2.6)
$$

Substitution of the first two of Eqs. (2.4) and of Eq. (2.3) in Eqs. (2.1) and (2.2) and retaining only those terms which contain ϵ^* to the power of one or less yields two sets of differential equations, one governing the static problem and the other the dynamic problem. These are

$$
\begin{aligned}
D\Delta\Delta\overline{w}^S - Q(F^S, \overline{w}^S) &= 0 \\
\Delta\Delta F^S &= -\frac{Eh}{2} Q(\overline{w}^S, \overline{w}^S)
\end{aligned}
\qquad (2.7)
$$

$$
\begin{aligned}
D\Delta\Delta\overline{w}^D - Q(F^S, \overline{w}^D) - Q(F^D, \overline{w}^S) &= \rho h \omega^2 \overline{w}^D \\
\Delta\Delta F^D &= -Eh Q(\overline{w}^S, \overline{w}^D)
\end{aligned}
\qquad (2.8)
$$

Since all detailed discussions of this plate are for a solid circular one of outside radius R, the problem is rephrased in terms of the polar coordinates. The static configuration has been shown (12) to be radially symmetric, but nonsymmetric dynamic configurations are permitted. Hence, all quantities are chosen in the following form* without significant loss

---

* Henceforth, unless otherwise noted, a summation symbol not having the summation limits specified is intended to be summed over n from 0 to ∞. The last term in connection with $\overline{v}^D$ has been added to accommodate a rigid rotation; this is discussed in more detail later on.

of generality:

$$\bar{u}^S = \bar{u}^S(r) \qquad \bar{u}^D = \sum \bar{u}_n^D(r) \cos n\Theta$$

$$\bar{v}^S = 0 \qquad \bar{v}^D = \sum \bar{v}_n^D(r) \sin n\Theta + (\tfrac{\gamma}{R})^2 B^*r$$

$$\bar{w}^S = \bar{w}^S(r) \qquad \bar{w}^D = \sum \bar{w}_n^D(r) \cos n\Theta \tag{2.9}$$

$$F^S = F^S(r) \qquad F^D = \sum F_n^D(r) \cos n\Theta$$

in which $\bar{u}$ and $\bar{v}$ represent the displacement components in the radial and tangential directions, respectively.

In order to render all pertinent quantities in these equations dimensionless, we let

$$x = \frac{r}{R}$$

$$U = \frac{R}{\gamma^2}\bar{u}^S \qquad u_n = \frac{R}{\gamma^2}\bar{u}_n^D$$

$$V = \frac{R}{\gamma^2}\bar{v}^S \qquad v_n = \frac{R}{\gamma^2}\bar{v}_n^D$$

$$W = \frac{\bar{w}^S}{\gamma} \qquad w^n = \frac{\bar{w}_n^D}{\gamma} \tag{2.10}$$

$$\Phi = \frac{F^S}{D} \qquad \phi^n = \frac{F_n^D}{D}$$

where $\gamma^2 = h^2/12(1-\nu^2)$.

By the use of the above expressions the differential equations for the static case become

$$\nabla^4 W - \frac{1}{x}(\Phi' W')' = 0 \tag{2.11a}$$

$$\nabla^4 \Phi = -\frac{1}{2x}(W'W')' \tag{2.11b}$$

in which $\nabla^2() = \frac{1}{x}[x(\ )']'$ and primes designate differentiation with respect to x.

The dynamic case for the n^{th} mode* is governed by

$$(\nabla^2 - \frac{n^2}{x^2})^2 w^n - \frac{1}{x}(\Phi' w^{n\prime})' + \frac{n^2}{x^2}\Phi'' w^n \qquad (2.12a)$$

$$- \frac{1}{x}(\phi^{n\prime} W')' + \frac{n^2}{x^2}\phi^n W'' = \mu^n w^n$$

$$(\nabla^2 - \frac{n^2}{x^2})^2 \phi^n = - [\frac{1}{x}(W' w^{n\prime})' - \frac{n^2}{x^2} W'' w^n] \qquad (2.12b)$$

where $$\mu^n = \frac{\rho h R^2 \omega_n^2}{D}$$

In order to state the boundary conditions we rewrite all of the quantities involved, including moments, membrane forces and strains in terms of suitable non-dimensional quantities. We thereby define:

$$N^S_{rr} = \frac{D}{R^2}\frac{1}{x}\Phi' = \frac{D}{R^2}T_{xx}$$

$$N^S_{\theta\theta} = \frac{D}{R^2}\Phi'' = \frac{D}{R^2}T_{\theta\theta}$$

$$N^D_{rr} = \frac{D}{R^2}\sum(\frac{1}{x}\phi^{n\prime} - \frac{n^2}{x^2}\phi^n)\cos n\theta = \frac{D}{R^2}\sum t^n_{xx}\cos n\theta$$

$$N^D_{\theta\theta} = \frac{D}{R^2}\sum \phi^{n\prime\prime}\cos n\theta = \frac{D}{R^2}\sum t^n_{\theta\theta}\cos n\theta \qquad (2.13)$$

$$N^D_{r\theta} = \frac{D}{R^2}\sum n\left(\frac{\phi^n}{x}\right)'\sin n\theta = \frac{D}{R^2}\sum t^n_{x\theta}\sin n\theta$$

Similarly the stress-strain relationships among the dimensionless quantities are:

$$\bar{e}^S_{xx} \equiv E_{xx} = T_{xx} - \nu T_{\theta\theta} = \frac{1}{x}\Phi' - \nu\Phi''$$

$$\bar{e}^S_{\theta\theta} \equiv E_{\theta\theta} = T_{\theta\theta} - \nu T_{xx} = \Phi'' - \frac{\nu}{x}\Phi'$$

$$e^n_{xx} = t^n_{xx} - \nu t^n_{\theta\theta} = \frac{1}{x}\phi^{n\prime} - \frac{n^2}{x^2}\phi^n - \nu\phi^{n\prime\prime}$$

$$e^n_{\theta\theta} = t^n_{\theta\theta} - \nu t^n_{xx} = \phi^{n\prime\prime} - \nu(\frac{1}{x}\phi^{n\prime} - \frac{n^2}{x^2}\phi^n) \qquad (2.14)$$

$$e^n_{x\theta} = \frac{E}{G}t^n_{x\theta} = 2(1+\nu)n(\frac{1}{x}\phi^{n\prime})'$$

---

* Separation into modes is possible because of the axial symmetry of the static solution.

The non-dimensionalized bending moments are defined as follows:

$$
\begin{aligned}
M^S_{rr} &= -D\left(\frac{\gamma}{R}\right)^2\left(W'' + \frac{\nu}{x} W'\right) \equiv D\left(\frac{\gamma}{R}\right)^2 M_{xx} \\
M^S_{\theta\theta} &= -D\left(\frac{\gamma}{R}\right)^2\left(\frac{1}{x} W' + \nu W''\right) \equiv D\left(\frac{\gamma}{R}\right)^2 M_{\theta\theta} \\
M^D_{rr} &= -D\left(\frac{\gamma}{R}\right)^2 \sum \left[w^{n''} + \nu\left(\frac{1}{x} w^{n'} - \frac{n^2}{x^2} w^n\right)\right] \cos n\theta \\
&\equiv D\left(\frac{\gamma}{R}\right)^2 \sum m^n_{xx} \cos n\theta \\
M^D_{\theta\theta} &= -D\left(\frac{\gamma}{R}\right)^2 \sum \left[\frac{1}{x} w^{n'} - \frac{n^2}{x^2} w^n + \nu w^{n''}\right] \cos n\theta \\
&\equiv D\left(\frac{\gamma}{R}\right)^2 \sum m^n_{\theta\theta} \cos n\theta \\
M^D_{r\theta} &= D(1-\nu)\left(\frac{\gamma}{R}\right)^2 \sum n\left(-\frac{w^{n'}}{x} + \frac{w^n}{x^2}\right) \sin n\theta \\
&\equiv D\left(\frac{\gamma}{R}\right)^2 \sum m^n_{x\theta} \sin n\theta
\end{aligned}
\tag{2.15}
$$

For the appropriate boundary conditions it is convenient to separate the static conditions from those associated with the vibration motion. Let the plate be simply supported at its circumference; this implies that both the deflection and the radial moment at the support are zero. In addition, either the membrane displacements or stresses must be specified. From a practical point of view the most realistic approach is to prescribe a given radial displacement for the static case (corresponding to a uniform increase in temperature with rigid supports) and rigid support for the dynamic case.

The boundary conditions governing the solution of the static problem are therefore

$$B_1(W) \equiv W(1) = 0 \tag{2.16}$$

$$B_2(W) \equiv \left(W'' + \frac{\nu}{x} W'\right)_{x=1} = 0 \tag{2.17}$$

and

$$\bar{u}^S(1) = -\lambda \bar{u}_E \tag{2.18}$$

Here $\bar{u}_E$ is the magnitude of the radial displacement which is required to cause the plate to buckle in the linear sense; thus the value of λ determines the extent to which the postbuckling domain is penetrated. This third condition is conveniently rephrased in terms of the stress function Since

$$\bar{u}^S = x\, e^S_{\theta\theta}$$

Eq. (2.18), after substitution of Eq. (2.14), leads to

$$B_3(\Phi) \equiv x[x(\frac{1}{x}\,\Phi')' + (1-\nu)(\frac{1}{x}\,\Phi')]_{x=1} = -\lambda U_E \qquad (2.19)$$

where $$U_E = \frac{R}{\gamma^2}\,\bar{u}_E$$

Following similar reasoning, the boundary conditions for the dynamic equations become

$$B_1(w^n) = 0 \qquad (2.20)$$

$$B_2(w^n) = 0 \qquad (2.21)$$

$$B_4(\phi^n) \equiv [\phi^{n''} - \nu(\frac{1}{x}\,\phi^{n'} - \frac{n^2}{x^2}\,\phi^n)]_{x=1} = 0 \qquad (2.22)$$

Eq. (2.22) implies the vanishing of the circumferential strain, $\epsilon_{\theta\theta}$ at the boundary. In general, a fourth boundary condition also associated with the vanishing of the displacements is needed. For the case $n = 0,1$ (which alone is considered in this paper), this represents, in part, a restriction on the permissible rigid body motion and therefore does not appear explicitly in the boundary conditions governing ϕ (i.e. it is utilized only when the membrane displacement components are computed). Conversely, the three boundary conditions Eqs. (2.20) to (2.22) are sufficient for determining w and ϕ if $n = 0,1$; this will become clear in what follows.

III. THE PERTURBATION SOLUTION

In recalling the differential equations and boundary conditions governing the problem, we consider first the static case. It is required to solve the following differential equations

$$\nabla^4 W - \frac{1}{x}(\Phi^{*\prime} W')' = 0 \tag{2.11a}$$

$$\nabla^4 \Phi^* = -\frac{1}{2x}(W'W')' \tag{2.11b}$$

and the associated boundary conditions*

$$B_1(W) = 0 \tag{2.16}$$

$$B_2(W) = 0 \tag{2.17}$$

$$B_3(\Phi^*) = -\lambda U_E \tag{2.18}$$

It is convenient here to partition the stress function such that

$$\Phi^* = \lambda\Phi_o + \Phi \tag{3.1}$$

where the function Φ_o satisfies the differential equation

$$\nabla^4 \Phi_o = 0 \tag{3.2}$$

and the boundary condition

$$B_3(\Phi_o) = -U_E \tag{3.3}$$

Consequently, the function Φ satisfies the differential equation

$$\nabla^4 \Phi = -\frac{1}{2x}(W'W')' \tag{3.4}$$

and the boundary condition

$$B_3(\Phi) = 0 \tag{3.5}$$

* The regularity requirements at the origin are being considered implicitly.

The first of these equations represents the usual problem of plane elasticity whose well-known solution for the solid disk is

$$\Phi_o = -\frac{Tx^2}{2} \tag{3.6}$$

where

$$T = \frac{U_E}{1-\nu}$$

and where U_E is found later on.

In order to proceed to the topic of this study, we first redevelop the solution previously obtained by Friedrichs and Stoker for the static condition. This is necessary because their results are presented in such a form as to make direct application to the present paper difficult. Moreover, they were able, through a change of the dependent variables, to reduce the order of the differential equations and to make them directly integrable, at least in part. This is not possible here since w appears explicitly in the dynamic equations as a result of the inertia term. Consequently, it is necessary to obtain a solution also in terms of W and Φ ; however, the procedure of Friedrichs and Stoker is being followed.

We assume the functions W, Φ, and λ to be expandable in a perturbation series:*

$$\begin{aligned} W &= \epsilon W_1 + \epsilon^3 W_3 + \epsilon^5 W_5 + \cdots \\ \Phi &= \epsilon^2 \Phi_2 + \epsilon^4 \Phi_4 + \epsilon^6 \Phi_6 + \cdots \\ \lambda &= \lambda_o + \epsilon^2 \lambda_2 + \epsilon^4 \lambda_4 + \cdots \end{aligned} \tag{3.7}$$

Here ϵ is the perturbation parameter which is chosen as a monotone increasing function whose direct significance will be fixed at a later point in the development. Substituting these perturbation expansions in the differential equations and boundary conditions and equating coefficients of like powers of ϵ yields a sequence of differential equations with associated boundary conditions. For example, associated with ϵ^0 is the equation whose solution is Φ_o .

For ϵ^1 the differential equation is, by virtue of Eq. (3.6),

$$L_1(W_1) \equiv \nabla^4 W_1 + \lambda_o T \frac{1}{x}(xW_1')' = 0 \tag{3.8}$$

and the boundary conditions are

$$B_1(W_1) = 0 \tag{3.9}$$

$$B_2(W_1) = 0 \tag{3.10}$$

* It can be shown that the other terms vanish.

This is the linear eigenvalue problem for the buckling of the plate subjected to compressive edge traction or displacement ($\lambda_o T$ represents the eigenvalue of the problem). Only the lowest eigenvalue and mode are of interest here; we therefore obtain, in conveniently normalized form,

$$W_1 = J_o(\alpha x) - J_o(\alpha) \tag{3.11}$$

in which α is the smallest root of

$$(1+\nu)J_1(\alpha) + \alpha J_2(\alpha) = 0 \tag{3.12}$$

and is related to the edge displacement by means of

$$\alpha^2 = T \tag{3.13}$$

after setting $\lambda_o = 1$ without loss of generality. This choice determines the meaning of ε . With Eqs. (3.13), the value of U_E is now

$$U_E = (1-\nu)T = (1-\nu)\alpha^2 \tag{3.14}$$

Higher roots of Eq. (3.12) lead to a complete set of functions of the type of Eq. (3.11); this set will be utilized for purposes of expansion in subsequent paragraphs.

For ϵ^2 the differential equation for Φ_2 is

$$\nabla^4\Phi_2 = -\frac{1}{2x}(W_1'W_1')' \tag{3.15}$$

with the associated boundary condition

$$B_3(\Phi_2) = 0. \tag{3.16}$$

This system is partially integrable, and upon using the regularity conditions at the origin the following expression in closed form is obtained through the use of familiar recursion relations (24):

$$\left(\frac{1}{x}\Phi_2'\right)' = -\frac{\alpha^2}{4x}\left[J_1^2(\alpha x) - J_o(\alpha x)\, J_2(\alpha x)\right] \tag{3.17}$$

Further integration of this expression seems impossible except by substitution of an infinite series. When this is done and another integration performed, the result is

$$\frac{1}{x}\,\Phi_2' = -\frac{\alpha^2}{4}\sum_{r=0}^{\infty}\frac{(-1)^r(1+2r)!\,(\frac{x}{2})^{2+2r}}{r!((2+r)!)^2(1+r)!} + C \qquad (3.18)$$

Another integration is not necessary since the function Φ_2 will not be needed explicitly. The constant C can be determined from B_3.

For ϵ^3 the differential equation governing W_3 is

$$L_1(W_3) = F_3(x) \equiv -\frac{1}{x}\lambda_2 T(xW_1')' + \frac{1}{x}(\Phi_2' W_1')' \qquad (3.19a)$$

with the associated boundary conditions

$$B_1(W_3) = 0 \qquad (3.19b)$$

$$B_2(W_3) = 0 \qquad (3.19c)$$

This inhomogeneous system is singular in the sense that the associated homogeneous system exhibits the nontrivial solution W_1. It therefore has either no solution or, if it has a solution, then this solution is not unique in that there can be added to it any arbitrary* multiple of W_1. In this latter case the right hand side must satisfy the orthogonality condition

$$\int_0^1 F_3(x)\,W_1 x\,dx = 0 \qquad (3.20)$$

which determines the coefficient

$$\lambda_2 = \frac{1}{T}\,\frac{\int_0^1 \Phi_2'(W_1')^2 dx}{\int_0^1 x(W_1')^2 dx} \qquad (3.21)$$

A particular solution of Eqs. (3.19) can now be constructed and the procedure continued to determine further perturbation coefficients and functions. However, as will become apparent in the sequel, there is no need to pursue the solution of the equilibrium problem beyond this point.

Regarding the vibration of the plate about the static buckled configuration and the associated equilibrium system of stresses as discussed in the

* While the choice is theoretically arbitrary, the specific value is selected on the basis of convenience of computation.

previous paragraphs, it is noted that the method of solution in the dynamic case is similar to the one used above and hence only the essential points are presented.

The equations governing the motion of the plate are Eqs. (2.12), which are presented again for the sake of convenience. By substitution of Eq. (3.6), they take the form

$$(\nabla^2 - \frac{n^2}{x^2})^2 w^n + \lambda_o T(\nabla^2 - \frac{n^2}{x^2}) w^n - \frac{1}{x}(\Phi' w^n)' + \frac{n^2}{x^2}\Phi'' w^n$$
$$- \frac{1}{x}(\phi^{n'} W')' + \frac{n^2}{x^2}\phi^n W'' = \mu^n w^n \quad (3.22a)$$

and
$$(\nabla^2 - \frac{n^2}{x^2})^2 \phi^n = -[\frac{1}{x}(W' w^{n'})' - \frac{n^2}{x^2} W'' w^n] \quad (3.22b)$$

subject to the boundary conditions

$$B_1(w^n) = 0 \quad (3.22c)$$
$$B_2(w^n) = 0 \quad (3.22d)$$
and
$$B_4(\phi^n) = 0 \quad (3.22e)$$

for $n = 0,1$.

The functions w, ϕ and μ are now expanded in perturbation series utilizing the same parameter ϵ as in the static case, that is

$$w^n = w_o^n + \epsilon^2 w_1^n + \cdots$$
$$\phi^n = \epsilon\phi_1^n + \epsilon^3\phi_3^n + \cdots$$
$$\mu^n = \mu_o + \epsilon^2\mu_2 + \cdots \quad (3.23)$$

Upon substitution of these perturbation expansions* in Eqs. (3.22), a new sequence of differential equations is obtained whose solution follows procedures analogous to those presented for the static case. It also becomes apparent that the partitioning of the stress function Φ* enhances the similarity further.

For ϵ^0 the differential equation is

$$L_2(w_o^n) \equiv (\nabla^2 - \frac{n^2}{x^2})^2 w_o^n + \lambda_o T(\nabla^2 - \frac{n^2}{x^2}) w_o^n - \mu_o^n w_o^n = 0 \quad (3.24a)$$

* The fact that w^n and μ^n are even expansions in ϵ and that ϕ^n is an odd expansion may be easily verified upon substitution in the relevant equations. For the sake of brevity these steps are omitted here.

with the associated boundary conditions

$$B_1(w_o^n) = 0 \tag{3.24b}$$

$$B_2(w_o^n) = 0 \tag{3.24c}$$

This linear eigenvalue problem admits an infinite set of eigenvalues and eigenfunctions μ^{nm} and w_o^{nm} ($m = 1,2,\ldots$) where the latter is given by

$$w_o^{nm} = J_n(\beta_2^{nm}x) - \frac{J_n(\beta_2^{nm})}{I_n(\beta_1^{nm})} \cdot I_n(\beta_1^{nm}x) \tag{3.25}$$

which satisfies Eq. (3.24b) in which I_n is the modified Bessel function. The characteristic equations is obtained in the usual fashion from the boundary conditions Eq. (3.24c):

$$\begin{aligned} &I_n(\beta_1^{nm})[(2n + 1 + \nu)\beta_2^{nm}J_{n+1}(\beta_2^{nm}) - (\beta_2^{nm})^2 J_{n+2}(\beta_2^{nm})] \\ &+ J_n(\beta_2^{nm})[(2n + 1 + \nu)\beta_1^{nm}I_{n+1}(\beta_1^{nm}) + (\beta_1^{nm})^2 I_{n+2}(\beta_1^{nm})] = 0 \end{aligned} \tag{3.26}$$

in which β_1^{nm} and β_2^{nm} are related to α and μ^{nm} by

$$\begin{aligned} (\beta_2^{nm})^2 - (\beta_1^{nm})^2 &= \alpha^2 \\ \mu_o^{nm} &= (\beta_1^{nm})^2(\beta_2^{nm})^2 \end{aligned} \tag{3.27}$$

The functions w_o^{nm} obey the orthogonality conditions

$$\int_0^1 w_o^{nm} w_o^{rs} \, x \, dx = 0 \qquad \mu_o^{nm} \neq \mu_o^{rs} \tag{3.28}$$

and

$$\int_0^1 [(\nabla^2 - \frac{n^2}{x^2})^2 w_o^{nm} + \lambda_o T(\nabla^2 - \frac{n^2}{x^2}) w_o^{nm}] w_o^{rs} \, x dx = 0 \tag{3.29}$$

It is noted that while n represents the number of nodal diameters, the index m is related to the number of nodal circles appearing in the vibration pattern of the plate. A similar system for the case of a clamped edge plate was solved by Federhofer in 1935. (6)

For ϵ^1 the differential equation governing the function ϕ_1^n is

$$\left(\nabla^2 - \frac{n^2}{x^2}\right)^2 \phi_1^n = -\frac{1}{x}\left(W_1' w_o^{n\,\prime}\right)' + \frac{n^2}{x^2} W_1'' w_o^n \qquad (3.30a)$$

with the associated boundary conditions

$$B_4(\phi_1^n) = 0 \qquad (3.30b)$$

For $n = 0$ the function ϕ_1^n can be established explicitly; for all other values of n it is necessary to resort to numerical integration.

For ϵ^2 the differential equation governing the deflection function w_2^{nm} is

$$L_2(w_2^{nm}) = \mu_2^{nm} w_o^{nm} + f_2^{nm}(x) \qquad (3.31a)$$

in which

$$f_2^{nm} = -\lambda_2 T\left(\nabla^2 - \frac{n^2}{x^2}\right) w_o^{nm} + \frac{1}{x}\left(\Phi_2' w_o^{nm\,\prime}\right)' - \frac{n^2}{x^2}\Phi_2'' w_o^{nm} + \frac{1}{x}\left(\phi_1^{nm\,\prime} W_1'\right)' - \frac{n^2}{x^2}\phi_1^{nm} W_1'' \qquad (3.31b)$$

and in which associated boundary conditions are

$$B_1(w_2^{nm}) = 0 \qquad (3.31c)$$

$$B_2(w_2^{nm}) = 0 \qquad (3.31d)$$

As before, $f_2(x)$ must satisfy an orthogonality condition if Eqs. (3.31) are to exhibit a solution; after some manipulation this leads to

$$\mu_2^{nm} = -\frac{\int_0^1 f_2^{nm}(x) w_o^{nm}\, x dx}{\int_0^1 w_o^{nm} w_o^{nm}\, x dx} \qquad (3.32)$$

For $n = 0$, numerical results are readily obtained. In particular the value of the rate of change of frequency with respect to the load parameter is desired in the neighborhood of the condition of linear buckling, that is

$$\lim_{\epsilon \to 0} \frac{d\mu}{d\lambda} = \lim_{\epsilon \to 0} \left(\frac{d\mu}{d\epsilon}\right) / \left(\frac{d\lambda}{d\epsilon}\right) = \mu_2/\lambda_2 . \qquad (3.33)$$

It is noted from Eqs. (3.24) that $\mu_0^{00} = 0$ and $w_0^{00} = W_1$, as expected. This in turn implies, by comparing Eqs. (3.30) with Eq. (3.15), that $\phi_1^{01} = 2\Phi_2$. When this is substituted in Eq. (3.32) and Eqs. (3.31b) and (3.21) are considered, it follows, after some integrations by parts and in view of the boundary and regularity conditions, that

$$\mu_2^{01} = 2\lambda_2 T \frac{\int_0^1 x(W_1')^2 dx}{\int_0^1 x(W_1)^2 dx} \tag{3.34}$$

which is readily evaluated. Thus, for $n = 0$ and $m = 1$,

$$\lim_{\epsilon \to 0} \frac{d\mu}{d\lambda} = 49.29 \tag{3.35}$$

It is therefore evident that when the critical buckling condition is reached and the lowest frequency of vibration is zero, and as the plate proceeds into the postbuckled condition, the square of the lowest frequency increases initially in proportion to the buckling parameter, this ratio being identified in Eq. (3.35). Further calculations with the perturbation method become exceedingly cumbersome and are abandoned in favor of the power series method of the next chapter. However, the present result is exact as ϵ approaches zero since higher order expansions vanish for this condition.

IV. THE POWER SERIES METHOD

Another possible means of solving the system of differential equations presented here is to develop the solution in terms of a power series. Again we borrow the results of Friedrichs and Stoker for the solution of the static problem. The phrasing is slightly different and some of the numerical computations represent minor variations of theirs.

In solving Eqs. (2.11) subject to the boundary conditions Eqs. (2.16), (2.17), and (2.19), we express the functions W and Φ in even power series of the coordinate x, that is,

$$W = \sum_{m=0}^{\infty} a_m x^{2m} \tag{4.1}$$

$$\Phi = \sum_{m=0}^{\infty} b_m x^{2m} \tag{4.2}$$

It can be shown, by substitution in Eqs. (2.11), that the coefficients a_0, a_1, b_0, and b_1 are, at this point, arbitrary and that all of the other coefficients of the two series are given by the following recursion relationships:

$$a_m = \frac{1}{2m^2(m-1)} \sum_{i=1}^{m-1} i(m-i)\, a_i b_{m-i} \tag{4.3}$$

$$m \geq 2$$

$$b_m = -\frac{1}{4m^2(m-1)} \sum_{i=1}^{m-1} i(m-i)\, a_i a_{m-i} \tag{4.4}$$

In terms of the coefficients in the power series, the previously stated boundary conditions become

$$\sum_{0}^{\infty} a_m = 0 \tag{4.5}$$

$$\sum_{1}^{\infty} m(2m - 1 + \nu)\, a_m = 0 \tag{4.6}$$

$$\sum_{1}^{\infty} m(2m - 1 - \nu)\, b_m = -\frac{\lambda U_E}{2} \tag{4.7}$$

Eq. (4.5) serves to determine the coefficient a_0 after the others have been computed. Eqs. (4.6) and (4.7) must be solved simultaneously for the values of a_1 and b_1. The value of the coefficient b_0 remains arbitrary, as might be expected since the stress function Φ can in general contain as much as an arbitrary linear function of the cartesian coordinates without affecting the stresses. Once the value of λ is specified, it is therefore possible to obtain all of the coefficients necessary to describe the complete solution for the static case.

The dynamic case is governed by Eqs. (2.12) and associated boundary conditions* Eqs. (2.20) to (2.22).

Again the solution of the differential equations may be expressed as power series in x. By the usual methods this leads to

$$w^n = x^n \sum_{m=0}^{\infty} c_m^{(n)} x^{2m} \tag{4.8}$$

and

$$\phi^n = x^n \sum_{m=0}^{\infty} d_m^{(n)} x^{2m} \tag{4.9}$$

The recursion relationships for all values of n that evolve from this system are, after dropping the superscripts for c and d,

$$\begin{aligned} c_m = \frac{1}{16m(m-1)(m+n)(m+n-1)} \Big\{ & \mu^{(n)} c_{m-2} \\ & + (2m+n-2) \sum_{i=1}^{m} 2i(n+2m-2i)(a_i d_{m-i} + b_i c_{m-i}) \\ & - n^2 \sum_{i=1}^{m} 2i(2i-1)(a_i d_{m-i} + b_i c_{m-i}) \Big\} \\ & \qquad m \geq 2 \end{aligned} \tag{4.10}$$

$$\begin{aligned} d_m = -\frac{1}{16m(m-1)(m+n)(m+n-1)} \Big\{ & (2m+n-2) \sum_{i=1}^{m} 2i(n+2m-2i)\, a_i c_{m-i} \\ & - n^2 \sum 2i(2i-1)\, a_i c_{m-i} \Big\} \end{aligned} \tag{4.11}$$

* For $n \geq 2$ the number of boundary conditions increases to four, as pointed out previously.

while the coefficients c_0, c_1, d_0, and d_1 remain, at this point, undetermined.

In terms of these power series, the boundary conditions for the dynamic problem assume the following form:

$$\sum_{m=0}^{\infty} c_m = 0 \tag{4.12}$$

$$\sum_{m=0}^{\infty} [(2m+n)(2m+n-1+\nu) - \nu n^2]c_m = 0 \tag{4.13}$$

$$\sum_{m=1}^{\infty} [(2m+n)(2m+n-1-\nu) + \nu n^2]d_m = 0 \tag{4.14}$$

It can be verified by examination of the recursion relation in the cases of $n = 0$ and $n = 1$ that d_0 does not appear in the problem and this is again plausible in view of the remark made in connection with b_0.

The boundary conditions thus lead to a system of homogeneous linear algebraic equations in the three unknown coefficients c_0, c_1, and d_1, as well as of the eigenvalue μ. The characteristic determinantal equation associated with this system is of formidable complexity and has therefore not been assembled in explicit form. Nor would any particular benefit be derived from this since, for the numerical solution obtained here, the appropriate algorithm can be stated in terms of the equations themselves.

The solution of the system of equations is effected on the digital computer. This is achieved by assigning consecutively arbitrary values, say (for example) unity to the coefficients c_0, c_1 and d_1. The remaining coefficients in the power series expansions are then determined by the recursion relationships. Since a generic term depends linearly upon c_0, c_1 and d_1, each coefficient in the power series is then the sum of three polynomials in μ, each of these polynomials being multiplied by the constants c_0, c_1 and d_1 respectively. When this is substituted into the boundary equations (4.12), (4.13) and (4.14), the resulting system of equations can be written in the form

$$[A]\{b\} = 0 \tag{4.15}$$

in which the elements of the 3 x 3 square matrix [A] are power series in μ, and {b} is the 3 x 1 column matrix whose elements are c_0, c_1, and d_1. If {b} is not to vanish trivially, it is necessary and sufficient that the characteristic equation

$$|A| = 0 \tag{4.16}$$

be satisfied. This determinant is equivalent to a power series in μ, whose roots represent the required eigenvalues.

For computational purposes this process must, of course, be truncated; this may be done by terminating the power series for w and ϕ when the value of a coefficient falls below a designated value. However, convergence of these coefficients alone is not necessarily indicative of final convergence of Eq. (4.16). In fact, truncation of the expansion series for w and ϕ not only reduces the elements of [A] to polynomials in μ, but it also affects the coefficients in these polynomials. This process itself is convergent in the present case. The resulting characteristic equation is a polynomial, whose lowest root is found on the computer by standard methods.

Generally, about sixteen terms in the power series produce sufficiently convergent terms in [A] and it suffices to take five or six terms in the characteristic equation to obtain the roots. For very large penetration of the postbuckling domain the size of the numbers involved and the number of terms required for the necessary accuracy requires a modification of this technique due to the limitations imposed by the available computer. This consists of an iteration procedure prior to the establishment of the final polynomial in μ. The matrix [A] is evaluated on the basis of trial values for μ, and Eq. (4.16) is then satisfied iteratively. Computing time of this procedure is approximately eight times longer than by the first method.*

*Weinitschke (23), who faced a similar computation in connection with a problem in shells, has used several power series expansions for different parts of the region. Matching them together, he has required fewer terms in each series for convergence.

V. RESULTS AND DISCUSSION

The degree of penetration of the postbuckling domain is measured by λ, the ratio of the actual edge displacement to that required for initial instability. Friedrichs and Stoker (12, 13) use a parameter λ_S based upon a stress ratio. For convenience the results here are expressed in terms of λ; Figs. 4 and 5 show the relation between the two parameters. The main result offered here is the relation between μ (the squared frequency parameter) and λ. Fig. 1 shows the relation for a symmetric mode ($n = 0$) and the first mode having a nodal diameter ($n = 1$). The details of this relation in the vicinity of $\lambda = 1$ (that is, near the point of initial instability) are given in Fig. 2. In Fig. 3 are shown the shapes of the modes of vibration. The data used in plotting Figures 1, 2 and 3 and further information are given in Table I.*

The manner in which μ increases with λ in the vicinity of initial instability for the symmetric mode as shown by the power series analysis is borne out by the perturbation analysis. From Fig. 2 the numerical value of the slope is fifty, which compares well with the results of Eq. (3.35), i.e. 49.29. It is interesting to observe that after some increase in λ the frequency of the nonsymmetric mode is lower than that of the symmetric mode. Examination of the modes for the symmetric case shows that at a value of λ between 13 and 19 a nodal circle appears for the lowest frequency. Near this value of λ the frequency of the axially symmetric mode increases less rapidly and eventually becomes again less than that for $n = 1$. This behavior of the frequencies is reasonable inasmuch as the nonsymmetric mode is essentially inextensional while the symmetric mode is initially extensional and consequently the frequencies of the symmetric mode become greater than those for $n = 1$. Upon the appearance of the nodal circle in the symmetric mode, this mode also becomes essentially inextensional; this may explain why the frequency falls again below that for $n = 1$.

Apparently the frequency reaches an asymptotic value as λ is increased. It should be noted, however, that for large values of λ the accuracy** of the results becomes less certain; moreover, the results themselves lose meaning since, in the limit, the plate is stretched as a membrane except for a narrow boundary layer at the edge, where also large bending stresses occur. Whether a plate can reach such a state is subject to question on practical grounds. The effect of initial imperfections, the onset of plastic yielding or secondary buckling, and several other questions make the theoretical idealized results appear somewhat academic for sufficiently large values of λ.

An apparently significant observation is that, at least within the range of the present computations, the frequency of vibration does not return to zero for $\lambda > 1$. This implies that the potential energy is positive definite and hence the buckled configuration is stable if only expansions up to the second power in the terms representing the additional neighboring deflections are included. It appears likely that this is true also in relation to higher modes. Consequently the experimentally observed phenomenon of secondary buckling [12] cannot be explained in terms of a simple branch point. Perhaps

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* All calculations are based upon the value of Poisson's ratio $\nu = .318$.

** "Accuracy" is defined here to be the ratio of the largest violation of boundary condition Eq. (2.20) to the maximum deflection. This ratio ranges from an optimum of 10^{-8} to a "poor" 10^{-4}.

there exists the possibility of a discontinuous snap-through to a position of lower potential energy. This becomes possible when the quadratic form introduced in Ref. (19) loses its positive definite character, as is indeed the case here. The problem of secondary buckling therefore shows great similarity with that of buckling of certain types of shells.

REFERENCES

1. Timoshenko, S. and Young, D. H. Vibration Problems in Engineering, 3rd Ed., New York, N. Y.: McGraw-Hill Book Company, 374-375.

2. Lurie, H. "Lateral Vibrations as Related to Structural Stability." J. Appl. Mech. 19, No. 2, 195-208.

3. Massonnet, Ch., "Les Relations entre les Modes Normaux de Vibration et la Stability des Systems Elastiques," Bulletin des Cours et des Laboratories d'Essais des Constructions du Genie Civil et d'Hydraulique Fluviale, Liege Universite, Centre d'etudes des recherches.

4. Timoshenko, S. Theory of Elastic Stability, Eng. Soc. Monograph New York, N. Y.: McGraw-Hill Book Company (1936) 367.

   Timoshenko, S. and Woinowsky-Krieger, Theory of Plates and Shells, 2nd Ed. New York, N. Y.: McGraw-Hill Book Company (1959)

5. Bryan, G. H. "Stability of a Plane Plate Under Thrusts in its Own Plane." Proc. of the London Math. Society, 22, (1891) 54.

6. Federhofer, K. "Biegungschwingungen der in ihrer Mittelebene belasteten Kreisplatte," Ing.-Archiv 6, (February 1935), 68-74.

7. Willers, F. A., "Eigenschwingungen gedruckter Kreisplatten," Zeitschrift fur angewandte Mathematik und Mechanik, 20 (1940) 37.

8. von Karman, T. "Festigkeitsprobleme im Maschinenbau," Encylkopadie der Mathematischen Wissenshaften, Vol. IV, 4, Teubner, Leipzig, 1910, 348-352.

9. Marguerre, K. "Die mittragende Breite der gedruckten Platte," NACA TN 833 (Luftfahrforschung, 14, 1937, No. 3, p. 121.)

10. Bisplinghoff, R. L. and Pian, T. H. H. "On the Vibrations of Thermally Buckled Bars and Plates," 9th International Congress for Applied Mechanics, 7, (1956) 307-318.

11. Way, S. "Bending of Circular Plates with Large Deflections," Trans. ASME, 56, (1934) 627.

12. Friedrichs, K. O. and Stoker, J. J. "Buckling of the Circular Plate Beyond the Critical Thrust," J. Appl. Mech. 9, (1942) 47-414.

13. Friedrichs, K. O. and Stoker, J. J. "The Nonlinear Boundary Value Problem of the Buckled Plate," Amer. J. of Math., 63, (1941) 839-888.

14. Bodner, S. R. "The Post Buckling Behavior of a Clamped Circular Plate," Quart. Appl. Math., 12, (1954) 397-401.

15. Bromberg, E. "Non-Linear Bending of a Circular Plate Under Normal Pressure," Comm. Pure and Appl. Math., 9. (1956) 633-659.

16. Keller, H. B. and Reiss, E. L. "Iterative Solutions for the Non-Linear Bending of Circular Plates," Comm. Pure and Appl. Math., 11 3, 1958.

Keller, H. B. and Reiss, E. L. "Non-Linear Bending of Circular Plates," Proc. of the 3rd U. S. National Congress of Applied Mechanics, 1958.

17. Alexeev, S. A. "The Postcritical Behavior of Flexible Elastic Plates," Prikladnaya Matematika i Mekhanika, 20, (1956) 673-679. (In Russian.)

18. Panov, D. U. and Feodossiev, V. I. "On the Equilibrium and Loss of Stability of Sloping Shells for Large Deflections," Prikladnaya Matematika i Mekhanika 12 (1948) 389. (In Russian.)

19. Masur, E. F. "On the Analysis of Buckled Plates," Proc. of the 3rd U. S. National Congress of Applied Mechanics, 1958.

20. Massonnet, Ch., "LeVoilement des Plaques Planes Sollicitees dans leur Plane," Final Report of the Third Congress of the International Association for Bridge and Structural Engineering, Liege, Belgium, September, 1948.

21. Reissner, E. "On Transverse Vibrations of Thin Shallow Shells," Quart. Appl. Math. 13 (1955) 169-176.

22. Naghdi, P. M. "On the General Problem of Elastokinetics in the Theory of Shallow Shells," Proc. of the Symposium on the Theory of Thin Elastic Shells, International Union of Theoretical and Applied Mechanics, Delft, (August 1959), 301-330.

23. Weinitschke, H. "On the Stability Problem for Shallow Spherical Shells," J. Math. Physics 38, No. 4, (1960) 209-231.

24. McLachlan, N. W. Bessel Functions for Engineers, 2nd Ed. London: Oxford University Press, 1955.

TABLE I

FREQUENCY PARAMETERS μ FOR n = 0, n = 1 FOR VARIOUS VALUES OF λ, λ_S

λ	λ_s	μ	
		n=0	n=1
1.00001	1.00000	0.000	131.410
1.00009	1.00001	0.004	131.412
1.00017	1.00003	0.009	131.414
1.00034	1.00005	0.017	131.419
1.00067	1.00010	0.033	131.427
1.00101	1.00015	0.050	131.435
1.00402	1.00059	0.198	131.511
1.00887	1.00129	0.437	131.632
1.01574	1.00229	0.775	131.804
1.02462	1.00359	1.213	132.026
1.03552	1.00517	1.750	132.299
1.04846	1.00705	2.387	132.623
1.06343	1.00923	3.124	132.998
1.08029	1.01168	3.954	133.420
1.09886	1.01437	4.869	133.886
1.41138	1.05927	20.258	141.735
1.98215	1.13931	48.371	156.161
2.91311	1.26498	94.298	179.920
4.15896	1.42499	155.952	212.109
4.42706	1.45835	169.243	219.088
6.31125	1.68353	262.640	268.598
7.11236	1.77494	302.177	289.862
9.57456	2.04258	421.501	355.833
11.25514	2.21547	499.317	401.278
13.86542	2.47114	610.406	472.235
19.60922	2.99070	794.910	629.174
29.17450	3.76396	948.097	886.156
33.88582	4.11459	988.571	1008.043
42.29802	4.70397	1035.067	1213.347
50.15607	5.22066	1061.999	1388.049
61.01744	5.89279	1091.762	1603.025
70.96683	6.47418	1122.798	1774.656

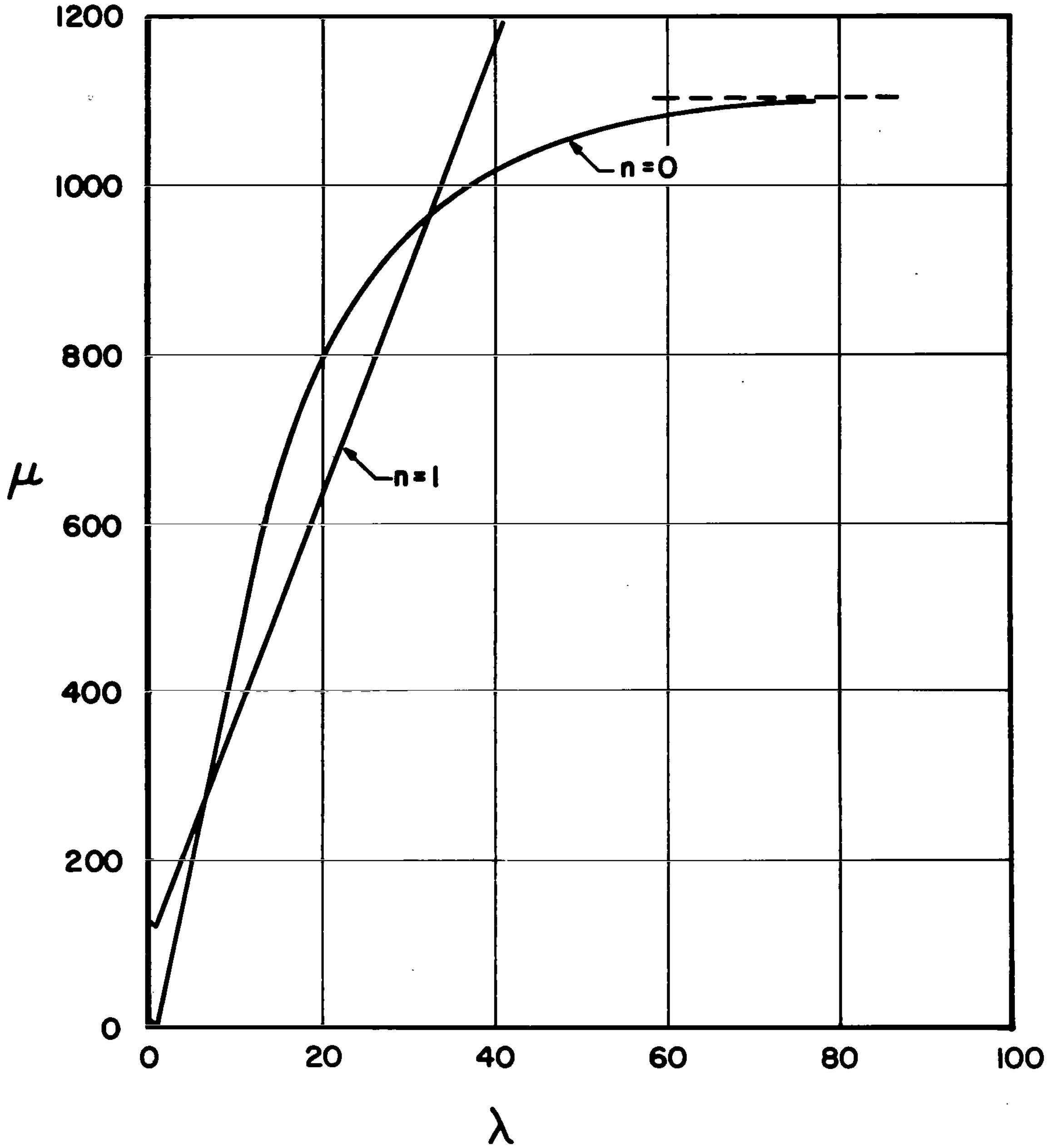

Figure 1. The Relation Between the Frequency Parameter μ and the Load Parameter λ.

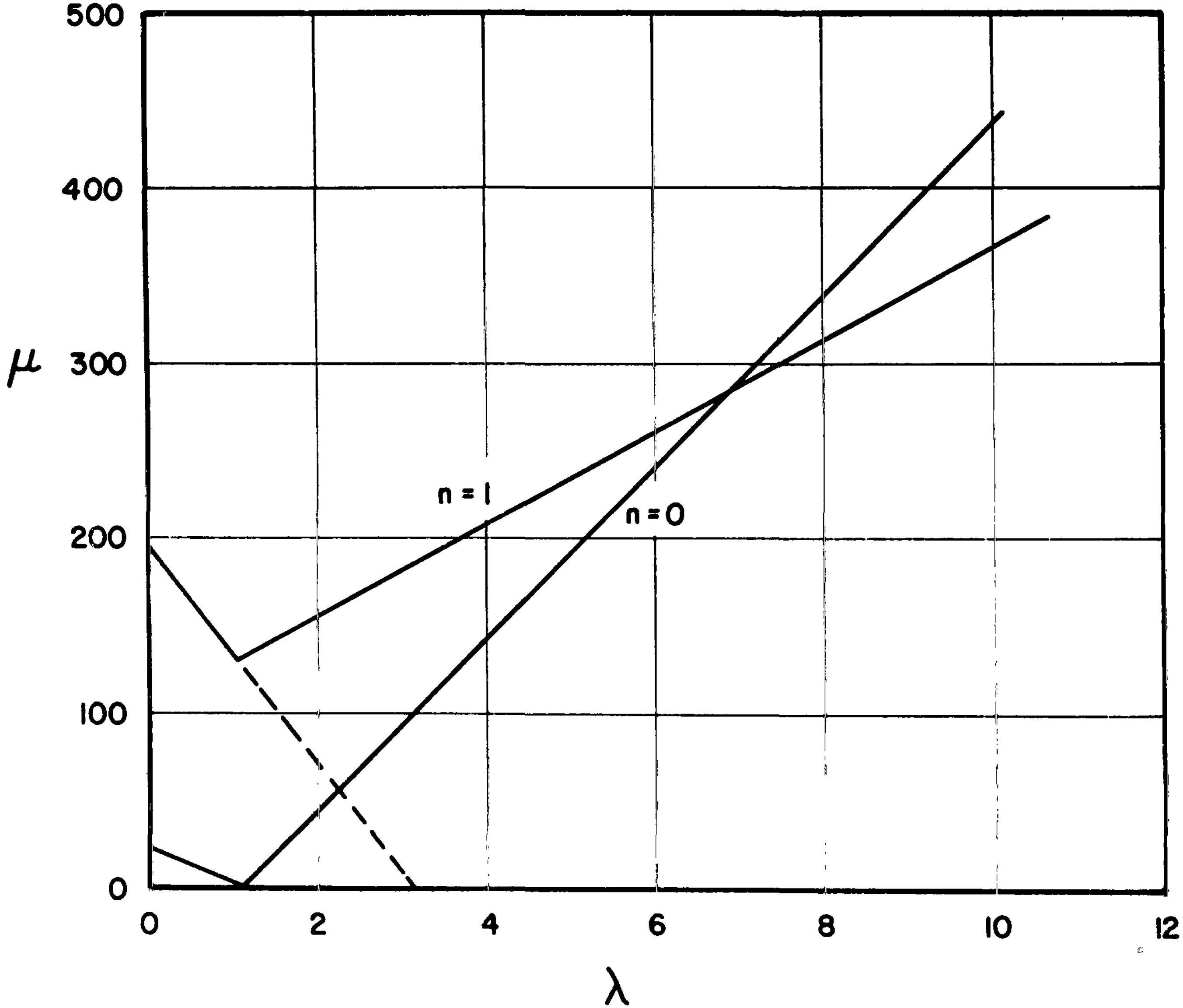

Figure 2. The Relation Between the Frequency Parameter μ and the Load Parameter λ for Small λ.

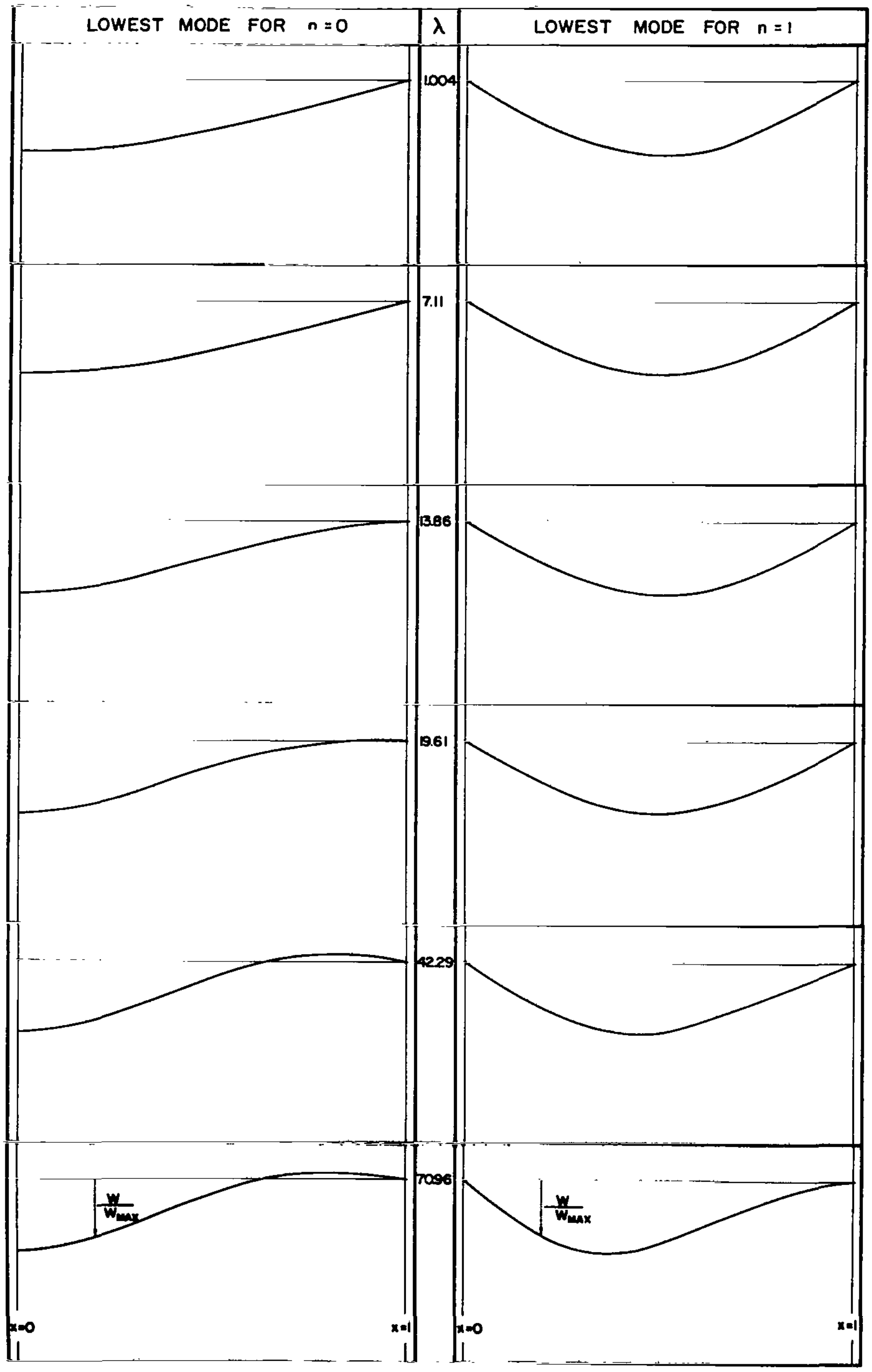

Figure 3. The Modes of Vibration Plotted for Half the Plate.

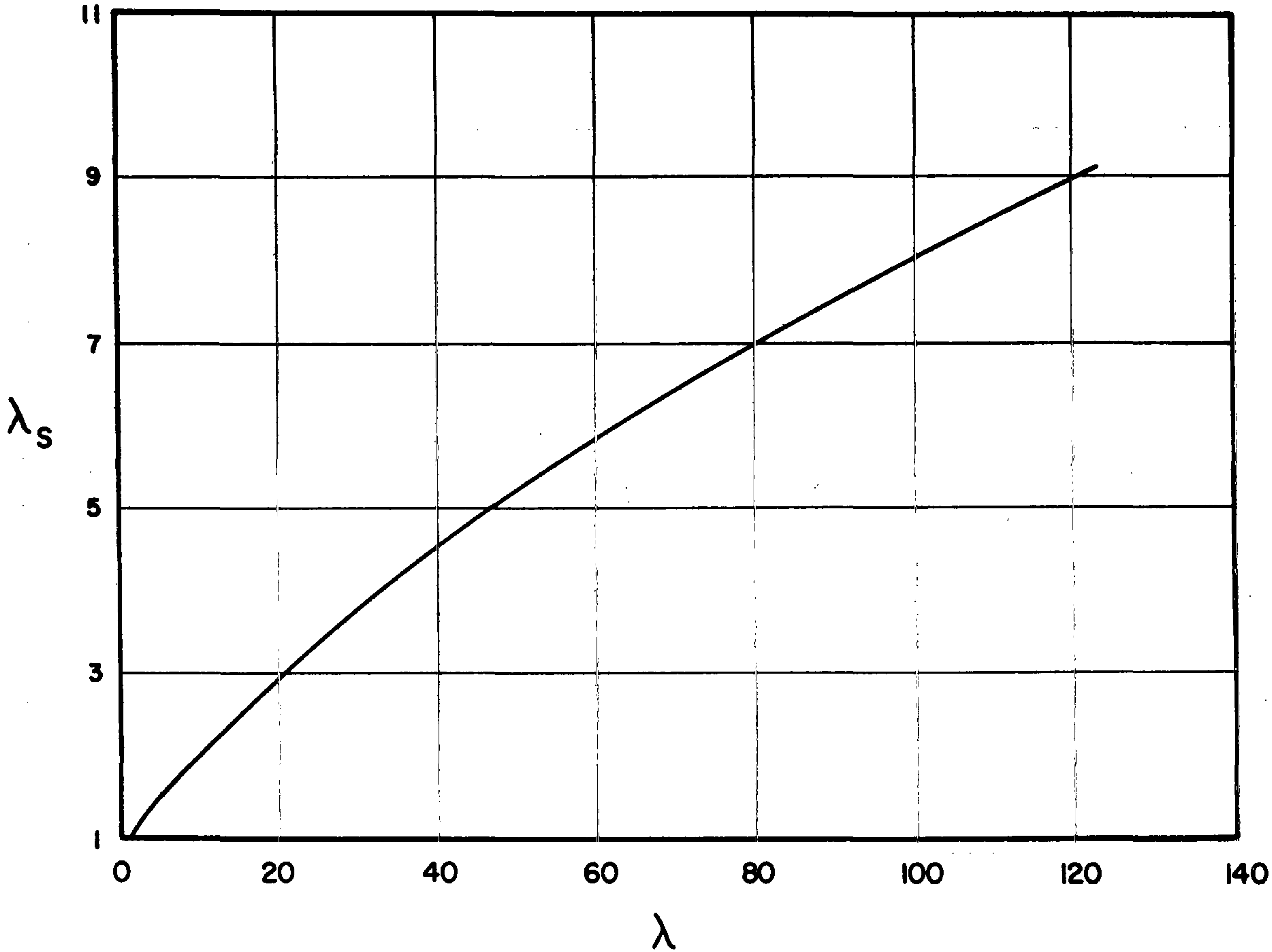

Figure 4. The Relation Between λ_S and λ.

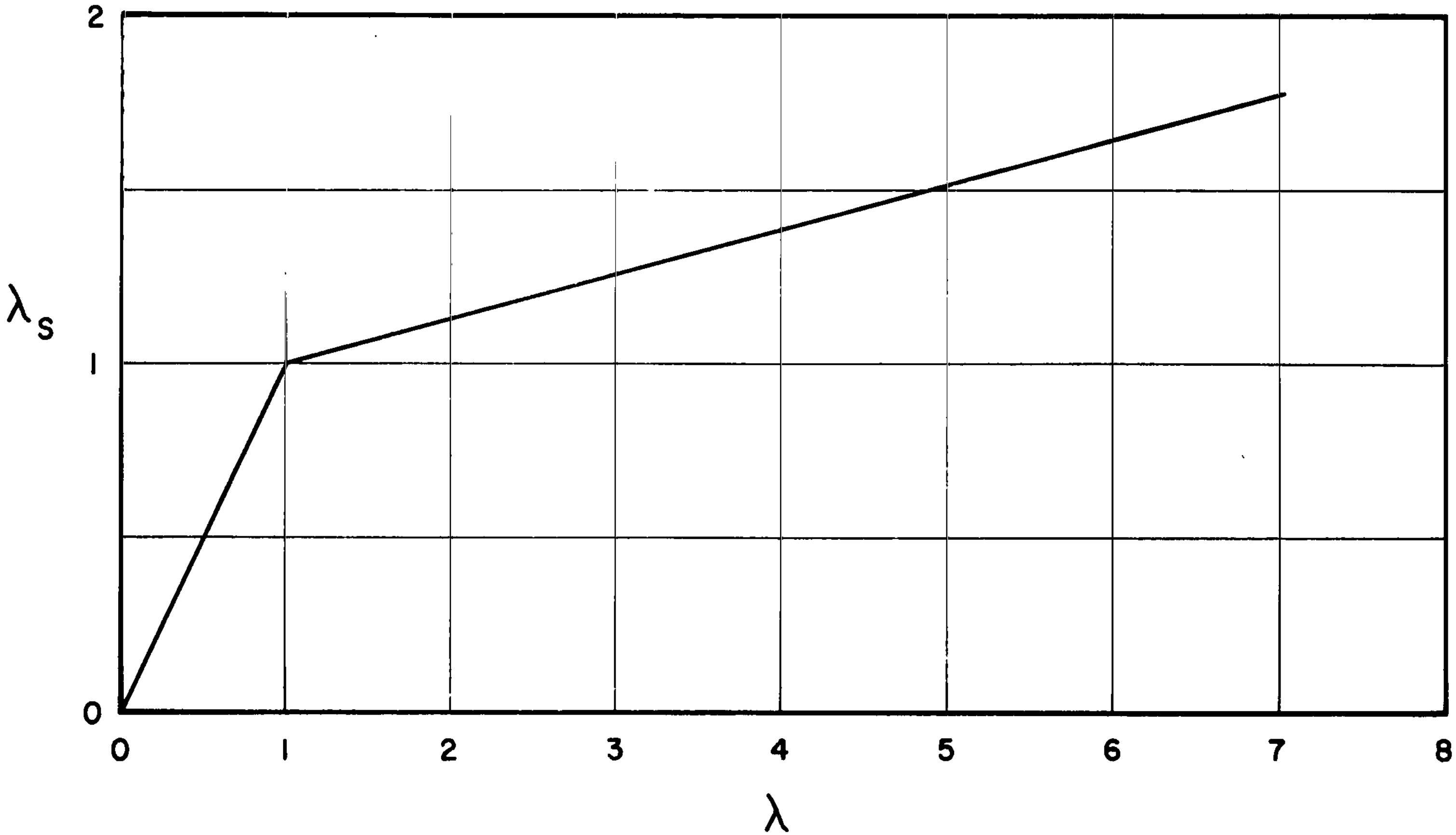

Figure 5. The Relation Between λ_s and λ for Small λ.

NASA-Langley, 1964 D-2245